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Lee waves in three-dimensional stratified flow

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The effect of a shallow isolated topography on a linearly stratified, three-dimensional, initially uniform flow in the x-direction is considered. The Green-function solution for the velocity disturbance due to this topography, which is equivalent to that due to a dipole at the origin, is shown to be without swirl, i.e. the velocity disturbance lies strictly in planes passing through the x-axis. Thus this disturbance can be described in terms of a stream function. The asymptotic forms of the wavelike portion of the stream function and the vertical displacement field are obtained. The latter is in agreement with the limited versions due to Crapper (1959). The Gaussian curvature of the zero-frequency dispersion surface is obtained analytically as a step in the stationary-phase calculation. The model is extended to determine the vertical displacement field for an arbitrary shallow topography far downstream. For topographies that are even functions of x and y it is shown that the details of the topography affect the displacement field only in the vicinity of the x-axis. Elsewhere, the amplitude of the displacement is proportional to the net volume of the topography.

1. Introduction

The study of stratified three-dimensional flows over a shallow topography is considerably less advanced than that of two-dimensional flows. A major difficulty lies in adequately describing this complex flow, for, apparently, a stream-function solution cannot be utilized as in the two-dimensional cases. Investigations (e.g. Wurtele 1957; Crapper 1959) have either been constrained to examine only the vertical velocity field or, (e.g. Smith 1980) have used the simplifying hydrostatic approximation.

At this point we can fairly state that the simplest problem, that of uniform, unbounded, stratified flow over an isolated topography, is not totally solved. We shall reexamine this problem and shall show that the velocity disturbance due to a dipole is without swirl and can be described in terms of a stream function. We shall obtain an asymptotic form of the wavelike portion of this disturbance and compare its form with that obtained by Crapper (1959). The structure of both the three-dimensional velocity and displacement fields due to the dipole will be examined. The model is then extended to determine the vertical displacement field due to an arbitrary topography.

We note that the model assumes an inviscid slightly disturbed flow. For a topography with sufficiently small slopes and heights everywhere, the small-disturbance theory should be valid. For high-Reynolds-number flows the effects of viscous forces should be limited to thin boundary layers and separated regions. For a shallow sloped topography it is expected that separated regions will be small and will have a limited effect on the external inviscid flow.

2. Formulation of the problem

We consider the small disturbance to a linearly stratified ($\bar{\rho}_\infty = \bar{\rho}_0(1-N^2z'/g)$) uniform flow of speed U in the positive x-direction due to a shallow topography $h'(x', y')$ superimposed on the $z'=0$ plane (figure 1). The dimensionless perturbation equations under the Boussinesq approximation are

$$\frac{\partial \boldsymbol{v}}{\partial x} = -\nabla p - \bar{\rho}\boldsymbol{i}_z, \tag{1}$$

$$\frac{\partial \bar{\rho}}{\partial x} = w, \tag{2}$$

$$\nabla \cdot \boldsymbol{v} = 0. \tag{3}$$

The boundary conditions are:

$$w(x, y, 0) = \frac{\partial h}{\partial x}(x, y), \tag{4}$$

$$\lim_{z\to\infty} \boldsymbol{v} = 0. \tag{5}$$

We note that if the disturbance is wavelike, condition (5) is supplemented by the requirement that the vertical component of the group-velocity vector be positive.

The previous equations have been non-dimensionalized using $L = U/N$ and a topographic reference height h_r, as follows:

$$\left.\begin{aligned} \boldsymbol{x} = \frac{\boldsymbol{x}'}{L}, \quad h = \frac{h'}{h_r}, \quad \boldsymbol{v} = \frac{\boldsymbol{v}'}{Nh_r}, \\ p = \frac{p'}{\bar{\rho}_0 UNh_r}, \quad \bar{\rho} = \frac{\bar{\rho}'}{\bar{\rho}_0 N^2 h_r/g}. \end{aligned}\right\} \tag{6}$$

As Crapper (1959) has shown, the range of the variable z can be extended from $0 \leqslant z < \infty$ to $-\infty < z < +\infty$ by replacing (3) with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2\frac{\partial h}{\partial x}(x, y)\,\delta(z). \tag{7}$$

As (1), (2) and (7) admit a solution for w that is an odd function of z, i.e. $w(x, y, 0^+) = -w(x, y, 0^-)$, when (7) is integrated in z from $z = 0^-$ to $z = 0^+$, condition (3) is recovered. Thus the solution to (1), (2) and (7) is equivalent to that of (1), (2) and (3).

We now demonstrate that the Green-function solution to (1), (2) and (7) is without swirl. We first let $h(x, y) = \delta(x)\,\delta(y)$, which is equivalent to a dipole singularity in (7). This means that

$$\iint h\,\mathrm{d}x\,\mathrm{d}y = \frac{\iint h'\,\mathrm{d}x'\,\mathrm{d}y'}{h_r L^2} = 1.$$

Hence $h_r = V'/L^2$, where V' is the dimensional volume of this localized topography. We now introduce the three-dimensional Fourier transforms of the dependent variable as follows.

$$(\mathscr{U}, \mathscr{V}, \mathscr{W}, \mathscr{P}, \mathscr{R}) = (2\pi)^{-\frac{3}{2}} \iiint_{-\infty}^{+\infty} (u, v, w, p, \bar{\rho})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}.$$

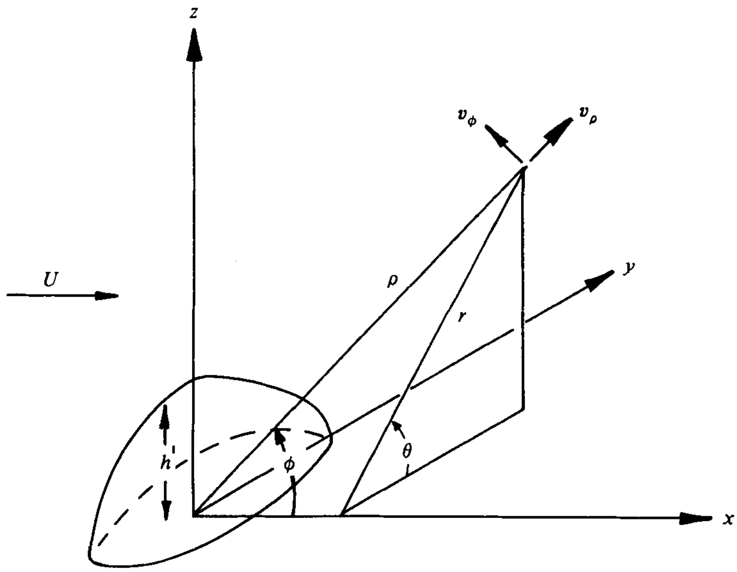

FIGURE 1. Geometry of the flow.

Taking the three-dimensional Fourier transforms of (1)–(3) we find that

$$(2\pi)^{\frac{3}{2}}\mathscr{V} = \frac{2k_1 k_2(k_1^2-1)}{k_1^2 k^2 - k_H^2}$$

and

$$(2\pi)^{\frac{3}{2}}\mathscr{W} = \frac{2k_3 k_1^3}{k_1^2 k^2 - k_H^2},$$

where

$$k^2 = k_1^2+k_2^2+k_3^2 \quad \text{and} \quad k_H^2 = k_1^2+k_2^2.$$

We can now show that

$$\frac{\partial \mathscr{V}}{\partial k_3} = \frac{\partial \mathscr{W}}{\partial k_2} = (2\pi)^{-\frac{3}{2}}\frac{4k_2 k_3 k_1^3(1-k_1^2)}{k_1^2 k^2 - k_H^2}. \tag{8}$$

As

$$\frac{\partial \mathscr{V}}{\partial k_3} = (2\pi)^{-\frac{3}{2}}\iiint_{-\infty}^{+\infty}(-izv)\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,d\boldsymbol{x}$$

and

$$\frac{\partial \mathscr{W}}{\partial k_2} = (2\pi)^{-\frac{3}{2}}\iiint_{-\infty}^{+\infty}(-iyw)\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,d\boldsymbol{x},$$

we conclude that $yw = zv$ and that the swirling velocity

$$v_\theta = (yw - zv)/r = 0.$$

This same result can be more laboriously obtained by inverting $\mathscr{V}$ and $\mathscr{W}$ with respect to k_3 through a residue calculation, then inverting with respect to k_2 using standard tables, and finally comparing the resulting integral with respect to k_1 (Appendix B). The lack of swirl means that the velocity disturbance lies in planes passing through the x-axis and can be expressed in terms of the axial component u and a radial component v_r, where

$$v = \frac{yv_r}{r}, \quad w = \frac{zv_r}{r},$$

where $r = (y^2+z^2)^{\frac{1}{2}}$.

Since the swirling component $v_\theta = 0$, this flow can be described by a stream function such that

$$\left.\begin{aligned} u = -\frac{1}{r}\frac{\partial\psi}{\partial r}, \quad v_r = \frac{1}{r}\frac{\partial\psi}{\partial x}, \\ v = \frac{y}{r^2}\frac{\partial\psi}{\partial x}, \quad w = \frac{z}{r^2}\frac{\partial\psi}{\partial x}. \end{aligned}\right\} \tag{9}$$

Crapper (1959) used as his dependent variable to the vertical particle displacement $\zeta(x,y,z)$, which is related to w by

$$w = \frac{\partial\zeta}{\partial x}. \tag{10}$$

Comparing (9) and (10), we find that

$$\psi = r^2\zeta/z. \tag{11}$$

Crapper obtained the solution for ζ as an integral over k_1 and evaluated the wavelike portion of his solution under various coordinate restraints. In §3 we shall obtain a more general form of the asymptotic ($\rho = (x^2+y^2+z^2)^{\frac{1}{2}} \gg 1$) wavelike solution of ζ and hence for the asymptotic velocity field.

3. The far field for the lee waves

To obtain the far-field solution for ζ and hence, through (11), ψ, we revert to a solution to (1)–(5) ($z \geqslant 0$). From (1)–(5) we can obtain, as did Crapper, the following relations:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+\frac{\partial^2\zeta}{\partial z^2}\right)+\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2} = 0, \tag{12}$$

$$\zeta(x,y,0) = \delta(x)\,\delta(y), \quad \lim_{z\to\infty}\zeta = 0. \tag{13a, b}$$

For the wavelike part of ζ we require that the vertical component of the group velocity be positive. The dispersion relation for small-amplitude internal gravity waves in a uniform flow in the x-direction is

$$(\omega' - Uk_1')^2 = N^2 k_{\mathrm{H}}'^2/k'^2, \tag{14}$$

where $k'^2 = k_1'^2+k_2'^2+k_3'^2$ and $k_{\mathrm{H}}'^2 = k_1'^2+k_2'^2$. Taking the partial derivative of this expression with respect to k_3' and letting $\omega' \to 0$, we find that

$$c_{gz} = \frac{\partial\omega'}{\partial k_3'} = \frac{N^2}{U}\left(\frac{k_3'}{k_1'}\right)\frac{k_{\mathrm{H}}'^2}{k'^4}.$$

For a positive value of c_{gz}, k_3' and k_1' must have the same sign.

To solve (12) and (13), we first Fourier transform these equations in x and y, and solve the resulting differential equation in z subject to (13) and the group-velocity requirement. We obtain, did Crapper,

$$\zeta = \frac{1}{4\pi^2}\iint_{-\infty}^{+\infty}\exp\left(\mathrm{i}(k_3(k_1,k_2)\,z+k_1x+k_2y)\right)\mathrm{d}k_1\,\mathrm{d}k_2, \tag{15a}$$

where
$$k_3 = \frac{k_{\mathrm{H}}(1-k_1^2)^{\frac{1}{2}}}{k_1} \quad \text{for} \quad |k_1| \leqslant 1, \tag{15b}$$

and
$$k_3 = \mathrm{i}k_{\mathrm{H}}(1-k_1^{-2})^{\frac{1}{2}} \quad \text{for} \quad |k_1| \geqslant 1. \tag{15c}$$

At this point Crapper inverted with respect to k_2 and obtained asymptotic expressions to ζ by considering various paths of integration in the complex k_1 plane, the paths taken depended on the relation between the independent variables. Here we proceed more generally via stationary-phase considerations. Our results will be more general than Crapper's, but correspond to his under his limiting relations. Also, because of (11) we also determine $\boldsymbol{v}$.

We shall seek an expression for ζ_{w}, the wavelike part of ζ which arises from the domain of integration $|k_1| \leqslant 1$, $-\infty < k_2 < \infty$. The major contribution to the double integral under the conditions that $z \neq 0$, $\rho \gg 1$ (Lighthill 1978) arise from the integral in the neighbourhood of those wavenumbers where the phase in the exponential term in (15a) is stationary. The phase is stationary when

$$\frac{\partial k_3}{\partial k_1} = -\frac{k_1^4 + k_2^2}{k_{\mathrm{H}} k_1^2 (1-k_1^2)^{\frac{1}{2}}} = -\frac{x}{z}, \quad \frac{\partial k_3}{\partial k_2} = \frac{k_2(1-k_1^2)^{\frac{1}{2}}}{k_{\mathrm{H}} k_1} = -\frac{y}{z}. \tag{16a,b}$$

We denote the solutions to these equations as k_1^* and k_2^* and $k_3^* = k_3(k_1^*, k_2^*)$. As ζ is clearly an even function of y, we consider only $y \geqslant 0$; we are only considering $z > 0$. Thus the wavelike response exists only for $x > 0$ and $k_1^* k_2^* < 0$. In determining the solution for $\boldsymbol{k}^*$ it is most convenient if we use spherical coordinates in physical space. We define $\rho = (x^2+y^2+z^2)^{\frac{1}{2}}$, $r = (y^2+z^2)^{\frac{1}{2}}$, $\cos\phi = x/\rho$, $\sin\phi = r/\rho$, $\sin\theta = z/r$ and $\cos\theta = y/r$ (figure 1). We find that there are two solutions for each $\boldsymbol{x}$, ${}^1\boldsymbol{k}^*$ and ${}^2\boldsymbol{k}^*$, with

$${}^1k_1^* = \sin\theta\cos\phi, \quad {}^1k_2^* = -\frac{\sin\theta\cos\theta\cos^2\phi}{\sin\phi}, \tag{17a,b}$$

$${}^1k_3^* = \frac{1-\sin^2\theta\cos^2\phi}{\sin\phi}, \quad {}^2\boldsymbol{k}^* = -{}^1\boldsymbol{k}^*. \tag{17c,d}$$

The contribution to the integral in the vicinity of ${}^2\boldsymbol{k}^*$ is the complex conjugate (c.c.) of that in the vicinity of ${}^1\boldsymbol{k}^*$ and serves to make w real. To continue with the stationary-phase calculation, the phase is expanded about $\boldsymbol{k}^*$ as follows:

$$\boldsymbol{k}\cdot\boldsymbol{x} = {}^1\boldsymbol{k}^*\cdot\boldsymbol{x} + \tfrac{1}{2}z \sum_{i=1}^{2}\sum_{j=1}^{2} k_{3,ij}(k_i - {}^1k_i^*)(k_j - {}^1k_j^*) + \ldots,$$

where

$$k_{3,ij} = \frac{\partial^2\, {}^1k_3^*}{\partial k_i \partial k_j}, \quad k_{3,i} = \frac{\partial\, {}^1k_3^*}{\partial k_i}, \quad i = 1,2.$$

Since $k_{3,ij}$ is a symmetric second-order tensor in horizontal-wavenumber space, the integral may be performed in the principal-axes system if this tensor, and we find that

$$\zeta_{\mathrm{w}} = \frac{1}{2\pi z}\frac{\exp(\mathrm{i}({}^1\boldsymbol{k}^*\cdot\boldsymbol{x} + {}^1\theta))}{|K_1 K_2|^{\frac{1}{2}}} + \text{c.c.}, \tag{18}$$

where K_1 and K_2 are the principal values of $k_{3,ij}$ and ${}^1\theta = \frac{1}{2}\pi$ if both K_1 and K_2 are positive, $-\frac{1}{2}\pi$ if both are negative, and ${}^1\theta = 0$ if they are of opposite sign. We shall show that K_1 and K_2 have opposite signs, so that ${}^1\theta = 0$, Further, the value of the determinant of a tensor that equals $K_1 K_2$ in the principal-axes system, is unchanged under a rotation so that $K_1 K_2 = k_{3,11} k_{3,22} - k_{3,12}^2$. Now, K_{G}, the Gaussian curvature of a surface specified by $k_3 = k_3(k_1, k_2)$ (see McConnell 1957), is given by

$$K_{\mathrm{G}} = \frac{K_1 K_2}{1 + k_{3,1}^2 + k_{3,2}^2}. \tag{19}$$

Using relations (16) in (19) and this result in (18), we find that

$$\zeta_{\mathrm{w}} = \frac{1}{2\pi}\frac{z}{\rho^2}\frac{\mathrm{e}^{\mathrm{i}\,{}^1\boldsymbol{k}\cdot\boldsymbol{x}}}{|K_{\mathrm{G}}|^{\frac{1}{2}}_{\boldsymbol{k}={}^1\boldsymbol{k}^*}} + \mathrm{c.c.} \tag{20}$$

The value of K_{G} for the surface given in (15b), which is, of course, the zero-frequency dispersion surface for internal gravity waves in a uniform flow, is given in Appendix A as

$$K_{\mathrm{G}} = -\frac{k_1^4(1-k_1^2)}{k_{\mathrm{H}}^2(k_1^4+k_2^2)}. \tag{21}$$

As we can see, $K_{\mathrm{G}} < 0$, so that $K_1 K_2 < 0$, and the phase shift is zero as noted earlier. Using ${}^1\boldsymbol{k}^*$ from (12) in (17) and (20) yields

$$\zeta_{\mathrm{w}} = \frac{1}{\pi}\frac{z}{\rho^2}\frac{\cos\phi(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}}{\sin^2\phi}\cos(\rho\sin\theta). \tag{22}$$

We note that this result is valid for $\rho \gg 1$ and $z \neq 0$ (hence $\phi \neq 0$). Using (11) for ψ_{w}, we find

$$\zeta_{\mathrm{w}} = \frac{1}{\pi}\frac{z}{r^2}\cos\phi(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}\cos(\rho\sin\theta), \tag{23}$$

$$\psi_{\mathrm{w}} = \frac{1}{\pi}\cos\phi(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}\cos(\rho\sin\theta). \tag{24}$$

A referee has pointed out that the expression for ζ_{w} agrees with that of Berkshire (1970).

For comparison purposes we can rewrite ζ_{w} in terms of x, y, z and r as

$$\zeta_{\mathrm{w}} = \frac{1}{\pi}\frac{z}{r^3}\frac{x}{x^2+r^2}(r^4+x^2y^2)^{\frac{1}{2}}\cos\left(\frac{xz}{r}\left(1+\frac{r^2}{x^2}\right)^{\frac{1}{2}}\right).$$

If $r^2 \ll x^2$

$$\zeta_{\mathrm{w}} \approx \frac{1}{\pi}\frac{z}{r^3 x}(r^4+x^2y^2)^{\frac{1}{2}}\cos\left(\frac{xz}{r}+\frac{rz}{2x}\right). \tag{25}$$

As noted earlier, Crapper first obtained an expression for ζ_{w} by inverting (15a) with respect to k_2. He then obtained expressions for ζ_{w} by considering various integration paths in the complex k_1 plane; the nature of the path depended on the relation between the x, y and z. He found for $x^2 \gg r^2$, $x^2y^2 \gg r^4$ that

$$\zeta_{\mathrm{w}} \sim \frac{1}{\pi}\frac{zy}{r^3}\cos\left(\frac{xz}{r}+\frac{rz}{2x}\right),$$

and for $x^2 \gg r^2$ and $z^2 \gg y^2$ that

$$\zeta_{\mathrm{w}} \sim \frac{1}{\pi}\frac{z}{r^3 x}(r^2z^2+x^2y^2)\cos\left(\frac{xz}{r}+\frac{rz}{2x}\right).$$

Our solution under the conditions that are imposed by Crapper yield identical results. This should give us some confidence in our results given the differing methods of computation employed here and in Crapper's work. We note that at $y = 0$

$$w = \frac{\partial\zeta_{\mathrm{w}}}{\partial x} = -\frac{1}{\pi x}\sin\left(x+\frac{z^2}{2x}\right).$$

Wurtele (1957) found an asymptotic expression for w which at $y = 0$ is proportional to $\cos(x+z^2/2x)/x$. Hence a discrepancy exists between his result and that of Crapper and ourselves.

Crapper ended his discussion of the dipole disturbance with his evaluation of ζ_w. Here, as we have shown that $\psi = r^2\zeta/z$, we can proceed to an evaluation of the velocity field. It is again most convenient to utilize the spherical components as $v_\theta = 0$. The radially outward component is v_ρ and the component v_ϕ is 90° to the left of v_ρ in a plane of constant θ (see figure 1). We find that

$$v_\rho = -\frac{1}{\rho^2 \sin\phi}\frac{\partial\psi}{\partial\phi} = \frac{(1-2\sin^2\theta\cos^2\phi)\cos(\rho\sin\theta)}{\pi\rho^2(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}}, \tag{26}$$

$$v_\phi = \frac{1}{\rho\sin\phi}\frac{\partial\psi}{\partial\rho} = -\frac{\sin\theta\cos\phi}{\pi\rho\sin\phi}(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}\sin(\rho\sin\theta). \tag{27}$$

We note that the dimensional amplitude of ψ is $NhrL^2 = NV'$. Stagnation points in the disturbance velocity field exist only if $\theta > \frac{1}{4}\pi$ and occur when $\cos\phi = 1/\sqrt{2}\sin\theta$ and $\rho\sin\theta = n\pi$.

The asymptotic velocity and vertical deflection field are discussed in §4.

4. A description of the asymptotic lee-wave field

We shall first discuss the velocity field in planes of constant θ. As we can see from (24), the flow is in annular cells occupying the region $0 < \phi \leqslant \frac{1}{2}\pi$ and $(n-\frac{1}{2})\pi \leqslant \rho\sin\theta \leqslant (n+\frac{1}{2})\pi$, where n is a large integer. The wavelength is $2\pi/\sin\theta$, and is a minimum in the (x, y)-plane. The flow is counterclockwise (clockwise) in each annular cell when n is odd (even). If $\theta < \frac{1}{4}\pi$ all streamlines start and end on the x-axis, while if $\theta > \frac{1}{4}\pi$ some streamlines are closed about the stagnation point. The flow for $\theta < \frac{1}{4}\pi$ and $\theta > \frac{1}{4}\pi$ is sketched in figure 2. We note that the solution is valid for $\rho \gg 1$, but is sketched for $\rho = O(1)$ for convenience.

The vertical deflection field was discussed in part by Crapper. It is best described in planes of constant z. We let

$$\left.\begin{aligned}
&\bar{\zeta} = \pi z\zeta_w, \quad \bar{x} = \frac{x}{z}, \quad \bar{y} = \frac{y}{z},\\
&\bar{r} = \frac{r}{z} = (1+\bar{y}^2)^{\frac{1}{2}},\\
&\bar{A} = \frac{\bar{x}(\bar{r}^4+\bar{x}^2(\bar{r}^2-1))^{\frac{1}{2}}}{\bar{r}^3(\bar{x}^2+\bar{r}^2)},\\
&\bar{\zeta} = \bar{A}\cos\left(z\left(1+\frac{\bar{x}^2}{\bar{r}^2}\right)^{\frac{1}{2}}\right).
\end{aligned}\right\} \tag{28}$$

Lines of constant phase in planes of constant z have

$$\frac{\bar{x}^2}{\bar{r}^2} = \bar{x}^{02} \quad \text{or} \quad \frac{\bar{x}^2}{\bar{x}^{02}} - \bar{y}^2 = 1,$$

and are hyperbolic. Here $\bar{x}^0$ is the value of $\bar{x}$ for a particular phase line at $\bar{y} = 0$. For large $\bar{y}$, $\bar{x} \to \bar{x}^0\bar{y}$. The amplitude $\bar{A}$ is a maximum (0.5) at $\bar{x} = 1, \bar{y} = 0$. Far downstream with $\bar{x}^2 \gg \bar{r}^2$, $\bar{A} \to \bar{A}_\infty(\bar{y})$, where

$$\bar{A}_\infty = \bar{y}(1+\bar{y}^2)^{-\frac{3}{2}}.$$

The amplitude far downstream has maximum value of $2/3\sqrt{3}$ (= 0.385) at $\bar{y} = 1/\sqrt{2}$. Contours of constant $\bar{A}$ are plotted in figure 3. For $0.5 > \bar{A} > 0.385$ the contours form closed loops which start and end at $\bar{y} = 0$. For $\bar{A} < 0.385$ the contours are in two

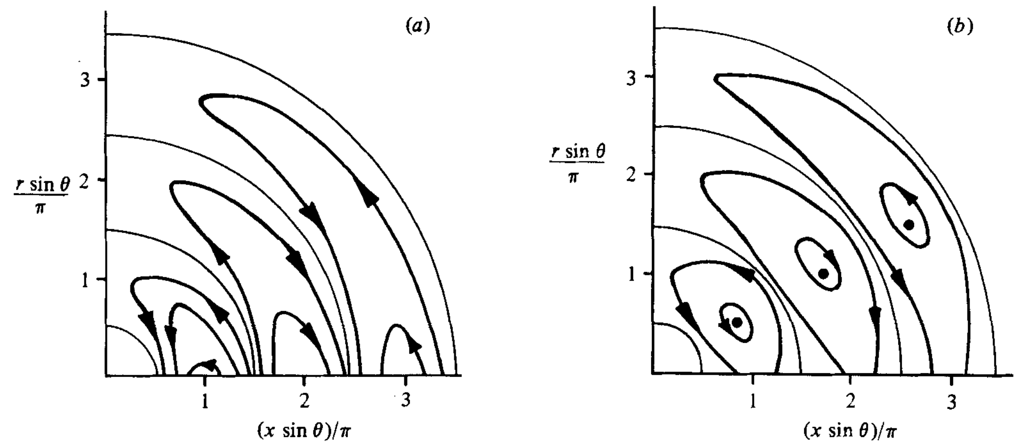

FIGURE 2. Streamline patterns for $\theta < \frac{1}{4}\pi$ (a) and $\theta > \frac{1}{4}\pi$ (b).

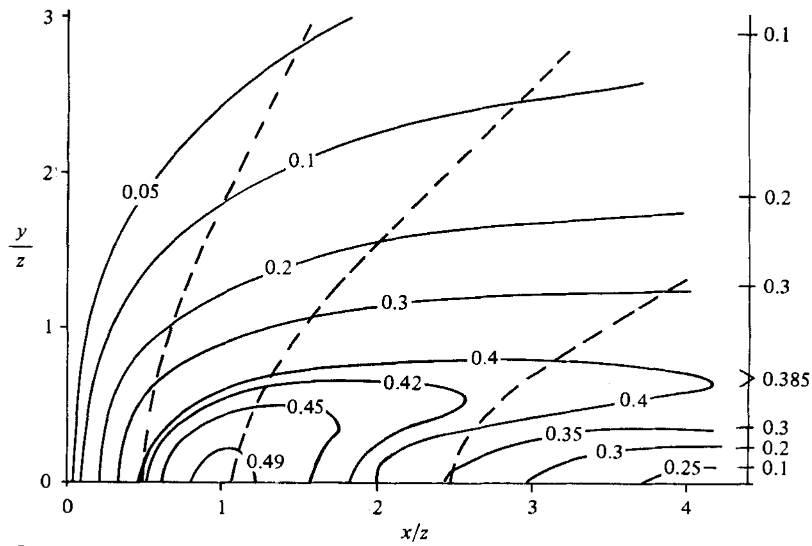

FIGURE 3. Lee-wave amplitude for the vertical displacement field. The solid lines are lines of constant amplitude while the dashed lines are of constant phase.

separate branches, each of which extends to infinity downstream. Also plotted in figure 3 are three lines of constant phase.

We can consider the variation of amplitude along lines of constant phase, i.e. we consider $\bar{A} = \bar{A}(\bar{x}^0, r)$. We see that

$$\bar{A}(\bar{x}^0, \bar{r}) = \frac{\bar{x}^0(\bar{r}^2(1+\bar{x}^{02}) - \bar{x}^{02})^{\frac{1}{2}}}{\bar{r}^3(1+\bar{x}^{02})}.$$

We can show that when $\bar{x}^0 < \sqrt{2}$, $\bar{A}$ is a maximum at $\bar{r} = 1$ ($\bar{y} = 0$). while for $\bar{x}^0 > \sqrt{2}$, $\bar{A}$ first increases as $\bar{y}$ increases from zero, and then decreases. The behaviour is seen in figure 3 as we move outward along lines of constant phase. We note that the location of the crests of the waves depends on both z and $\bar{x}^0$, and occur when

$$z(1+\bar{x}^{02})^{\frac{1}{2}} = 2\pi n, \quad n = 1, 2, \ldots.$$

The largest wave amplitude ζ'_w at a crest occurs at $z = x = \sqrt{2}\pi$, $y = 0$, and equals $hr/2\pi z = (1/2\sqrt{2}\pi^2)\,hr$.

5. Limitations and extensions of the theory

Before proceeding to a discussion of the extensions of the theory to a generalized shallow topography, let us recall some of the limitations that have been imposed. We have assumed that the flow is unbounded and has uniform speed and stability far upstream. Under these conditions, the Green-function, or dipole, solution shows no swirl. This flow can be described in terms of a stream function which has been evaluated far downstream. The presence of a vertical shear will create swirl and hence eliminate a streamfunction description. For a discussion of the effects of shear see Sharman & Wurtele (1983).

The presence of rigid horizontal boundaries located at $z' = \pm H$ will create swirl but can be accommodated by the theory by using image dipole solutions along the z-axis at $z' = \pm 2mH$ ($m = 1, 2, \ldots$). As the velocity associated with the dipole falls off with z as $(z^2+y^2)^{-1}$ at a fixed x, we can show that the effect of the image series when compared with the original dipole is $O(U/NH)$ away from the boundaries. Thus if $z' \ll H$ and $U/NH \ll 1$ the description of the flow given here is valid.

We now proceed with the extension of the theory to our arbitrary shallow topography. An arbitrary shallow topography $h'(x', y')$ with horizontal extents L_x and L_y in the x' and y' directions, and a double Fourier transform $H'(k_1', k_2')$, can be accommodated by inserting $H'(\boldsymbol{k}')/H'(0) = |H'(\boldsymbol{k}')/H(0)| \exp(\mathrm{i}\gamma(\boldsymbol{k}'))$ into the integrand on the right-hand side of (15a) and interpreting V' as the net volume of the topography ($= 2\pi H'(0,0)$). For $\rho' \gg L_x, L_y, U/N$ a stationary-phase calculation can be performed with the wavenumber of stationary phase as well as the Gaussian curvature of the dispersion surface calculate as before. We obtain for the dimensional vertical deflection,

$$\zeta_{\mathrm{w}}' = \frac{V'}{2\pi L}\left|\frac{H'(\boldsymbol{k}'^*)}{H'(0)}\right| \frac{\sin\theta\cos\phi}{r'}(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}\cos(\rho'\sin\theta/L+\gamma^*), \tag{29}$$

where
$$k_1'^* = \frac{\sin\theta\cos\phi}{L}, \quad k_2'^* = -\frac{\sin\theta\cos\theta\cos^2\phi}{L\sin\phi},$$

$$L = \frac{U}{N}, \quad r' = (y'^2+z'^2)^{\frac{1}{2}}, \quad \rho' = (x'^2+r'^2)^{\frac{1}{2}},$$

$$\sin\theta = \frac{z'}{r'}, \quad \cos\phi = \frac{x'}{\phi'}, \quad \gamma^* = \gamma(\boldsymbol{k}'^*),$$

and
$$H'(\boldsymbol{k}') = \frac{1}{2\pi}\iint_{-\infty}^{+\infty} h'(x', y')\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{x}'}\,\mathrm{d}x'\,\mathrm{d}y'.$$

Note that as $r' \to 0$ with θ fixed, $|k_2^*| \to \infty$, $H(k_2^*) \to 0$, and no singularity in ζ_{w}' exists.

For topographies that are even functions of x' and y', $H'(k_1', k_2')$ is real and we have

$$\zeta_{\mathrm{w}}' = \frac{V'}{2\pi L}\frac{H'(\boldsymbol{k}^*)}{H'(0)}\frac{\sin\theta\cos\phi}{r'}(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}\cos(\rho'\sin\theta/L). \tag{30}$$

We note that as $\theta \to 0$ with $\sin\phi \neq 0$, or $\cos\phi \to 0$, $|\boldsymbol{k}^*| \to 0$ and $H'(\boldsymbol{k}^*) \to H'(0)$. In these regions, the details of the topography are relatively unimportant. On the other hand, if $x'/z' \gg 1$, $|k_2^*| \to 0.385(x'/z')/L$ at $y'/z' = \sqrt{\frac{1}{2}}$, while $|k_2^*| = 0$ at $y' = 0$ and as $y'/z' \to \infty$. Thus the cross-flow structure of the topography is quite important between $y' = 0$ and $y' \sim (L_y/L)(z'x')^{\frac{1}{2}}$ (for large x'/z' and $y' > z'$, $|k_2^*|\,L_y \sim L_y z'x'/y'2L$). Now $|k_1^* L_x| \leqslant L_x/L$ at $y' = 0$ and decreases as y'/z' increases. Thus the details of the structure of the topography in the x' direction is important in the far field

($|k_1^* L_x| = O(1)$) only if $L_x/L \geqslant 1$, and then only when y'/z' is not large. The region near the x' axis far downstream is thus the region where the details of the structure of the topography is most pronounced; elsewhere the net volume of the topography determines the amplitude of deflections.

6. Summary and conclusions

We have examined the uniform stratified flow past a localized topography and have shown that the velocity disturbance produced by this dipole singularity is without swirl. Because of this, the velocity field can be described in terms of gradients of a streamfunction. The asymptotic solution for the lee-wave portion of both vertical displacement and streamfunction fields were obtained. The latter was more general, but consistent with that of Crapper (1959). The vertical displacement field far downstream for an arbitrary shallow topography was then determined. Detailed topographic dependence was shown to be confined to a region near the x-axis.

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Appendix A. The Gaussian curvature of the dispersion surface

The surface in wavenumber space given in (15*b*) can also be written as

$$\frac{|k_1|}{k_{\mathrm{H}}} = \frac{1}{k}, \qquad \text{(A 1)}$$

where $k_{\mathrm{H}} = (k_1^2 + k_2^2)^{\frac{1}{2}}$ and $k = (k_3^2 + k_{\mathrm{H}}^2)^{\frac{1}{2}}$. This surface can best be described in spherical coordinates with u_1, the angle between the wavenumber vector and the k_3 axis, and u_2, the angle between the horizontal projection of this vector and the k_1 axis. Thus

$$\left. \begin{array}{ll} k_1 = (k \sin u_1) \cos u_2, & k_2 = (k \sin u_1) \sin u_2, \\ k_3 = k \cos u_1, & k_{\mathrm{H}} = k \sin u_1. \end{array} \right\} \qquad \text{(A 2)}$$

Equation (A 1) can be rewritten as

$$k = 1/|\cos u_2|. \qquad \text{(A 3)}$$

Because of the symmetry in this surface we consider $0 \leqslant u_1, u_2 \leqslant \frac{1}{2}\pi$. Using (A 3) and (A 2) we have

$$\left. \begin{array}{ll} k_1 = \sin u_1, & k_2 = \sin u_1 \tan u_2, \\ k_3 = \cos u_1 \sec u_2, & k_{\mathrm{H}} = \sin u_1 \sec u_2. \end{array} \right\} \qquad \text{(A 4)}$$

The surface is completely defined in terms of the values of u_1 and u_2. In obtaining K_{G}, the Gaussian curvature of this surface, we make use of relations given in McConnell (1957, chap. 14, §§3 and 9). The metric tensor of the surface is given by

$$a_{\alpha\beta} = \sum_{i=1}^{3} \frac{\partial k_i}{\partial u_\alpha} \frac{\partial k_i}{\partial u_\beta}, \quad \alpha, \beta = 1, 2.$$

Using (A 4) in this expression we find that

$$\left. \begin{array}{l} a_{11} = \sec^2 u_2, \\ a_{22} = \sec^4 u_2 (1 - \cos^2 u_1 \cos^2 u_2), \\ a_{12} = a_{21} = 0. \end{array} \right\} \qquad \text{(A 5)}$$

We note that the surface metric tensor is composed of the various dot products of $\partial \boldsymbol{k}/\partial u_1$, $\partial \boldsymbol{k}/\partial u_2$ which are vectors tangent to the surface in planes of constant u_2 and u_1 respectively. As $\partial k/\partial u_1 = 0$, the vector $\partial \boldsymbol{k}/\partial u_1$ ties strictly in the direction of increasing u_1 while $\partial \boldsymbol{k}/\partial u_2$ has its components in the radial and increasing u_2 directions. These two vectors are thus orthogonal and $a_{12} = 0$. For orthogonal coordinates,

$$K_G = -\frac{1}{2a^{\frac{1}{2}}}\left(\frac{\partial}{\partial u_1}\left(\frac{1}{a^{\frac{1}{2}}}\frac{\partial a_{22}}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{1}{a^{\frac{1}{2}}}\frac{\partial a_{11}}{\partial u_2}\right)\right). \tag{A 6}$$

where $a = a_{11}a_{22}$. Using (A 5) in (A 6), we find that

$$K_G = -\frac{\cos^4 u_2 \cos^2 u_1}{1-\cos^2 u_1 \cos^2 u_2}. \tag{A 7}$$

Using (A 4), we find that

$$K_G = -\frac{k_1^4(1-k_1^2)}{k_H^2(k_1^4+k_2^2)}. \tag{A 8}$$

Appendix B. The lack of swirl

The three dimensional Fourier transforms of v and w may be inverted with respect to k_3 via a residue calculation taking into account the radiation condition, and we find that

$$v(x,y,z) = \frac{\mathrm{i}}{4\pi^2}\int_{-\infty}^{+\infty}(\operatorname{sign} k_1)\,(k_1^2-1)^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}k_1 x}\int_{-\infty}^{+\infty}\frac{k_2}{\mathrm{i}k_H}\mathrm{e}^{-k_2 y - Ck_H}\,\mathrm{d}k_2$$

and

$$w(x,y,z) = \frac{\mathrm{i}}{4\pi^2}\int_{-\infty}^{+\infty}k_1\,\mathrm{e}^{\mathrm{i}k_1 x}\,\mathrm{d}k_1\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}k_2 y - Ck_H}\,\mathrm{d}k_2,$$

where

$$k_H = (k_1^2+k_2^2)^{\frac{1}{2}},$$

$$C = \frac{z(k_1^2-1)^{\frac{1}{2}}}{|k_1|} \quad \text{for} \quad |k_1| > 1$$

and

$$C = z\left(\frac{\epsilon}{k_1^2(1-k_1^2)^{\frac{1}{2}}}-\frac{\mathrm{i}(1-k_1^2)^{\frac{1}{2}}}{k_1}\right) \quad \text{for} \quad |k_1| \leqslant 1.$$

The real part of C for $|k_1| < 1$ arises if the dipole is viewed as being 'slowly switched on', i.e. if it is assigned a temporal variation as $\mathrm{e}^{\epsilon t}$, which vanishes as $t \to -\infty$ (see Crapper 1959). As $k_2\mathrm{e}^{-Ck_H}$ is an odd function and e^{-Ck_H} is an even function of k_2, the integrals over this variable in the above expressions may be taken over the interval 0 to infinity, and the results, which are standard, lead to the following integrals over k_1 as $\epsilon \to 0$:

$$\frac{v}{y} = \frac{w}{z} = \frac{\mathrm{i}}{2\pi^2}\int_{-\infty}^{+\infty}k_1|k_1|\,(k_1^2-1)^{\frac{1}{2}}\frac{K_1((r^2k_1^2-z^2)^{\frac{1}{2}})}{(r^2k_1^2-z^2)^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}k_1 x}\,\mathrm{d}k_1.$$

Hence $v/y = w/z$ or $zv = yw$, as shown earlier. We note that the expression for w is simply the partial derivative with respect to x of Crapper's solution for ζ_w.

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